Discrete Mathematics

NSM 6617

Class Notes

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Typical Problems in Discrete Mathematics

Which version of a lottery gives the best odds?

How many paths are there from a point to another in a network? What is the shortest path?

Given a logical argument, determine if it is valid.

Find a formula for the sum $1 + 3 + 5 + \ldots + 2n - 1$ and prove that your formula is correct.

What is the minimum number of colors needed to color a diagram so that no two adjacent areas have the same color?

Prove that there are an infinite number of primes.
Prove that there are an infinite number of twin primes.

Determine the number of isomers of C_nH_{2n+2}.

Find a solution to Instant Insanity.

Counting Principles

1. Addition Principle If a first task can be done in n ways and a second task can be done in m ways, and if these tasks can not be done at the same time, then there are n + m ways to do either task.

Example A manager must choose an employee from one of two departments which contain 35 and 27 people respectively. How many people can he choose from?

Example A student can choose a project from one of three lists. The three lists contain 30, 40 and 50 possible projects respectively. How many possible projects are there to choose from?

2. Multiplication Principle Suppose that a procedure can be divided into two steps such that the first step can be performed n ways and the second step can be performed m ways. Then the number of different ways to perform the entire procedure is nm.

Example Joe must travel from NY to Chicago to LA. He has identified 5 ways to go from NY to Chicago and 3 ways to go from Chicago to LA. How many ways can his trip be made?

Example The drama club of Dowling is holding tryouts for a play. With 6 men and 8 women auditioning for the leading male and female roles. How many ways are there to cast the leading couple?

Example How many ways are there to arrange the letters a, b, c and d?

Example How many bit strings of length 4 are there? Can you list them all?

Example How many ways are there to form a 3-letter sequence using the letters a, b, c, d, e and f:

(a) without repetition?

(b) with repetition?

(c) without repetition that contain an e?

(d) without repetition that contain an e?

3. A <u>permutation</u> of a set is any ordering of the set.

There are exactly n! permutations of a set with n distinct objects.

Example How many permutations are there of {a,b,c,d,e}?

Solution 5! = 120

Example List all permutations of {a,b,c}.

Example Eight horses have entered a race. If a spectator were to select 3 different horses at random to bet on for win (first palce), place (second place) and show (third place), how likely is he to be correct?

4. The number of permutations of length r from a set of size n is

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)...(n-r+1)$$

Example How many permutations of length 3 are there of {a,b,c,d,e}?

Solution (5)(4)(3) = 60.

5. If there are n objects with n_1 objects of type 1, n_2 objects of type 2, and so on, with $n = n_1 + n_2 + \ldots + n_r$, then the number of arrangements (permutations) of the n objects is given by

$$\frac{n!}{n_1!n_2!\cdots n_r!}.$$

Example How many permutations are there of the word MASSAPEQUA?

Example Determine the number of staircase paths in the xy plane from (2,1) to (7,4). Assume that each path is allowed to move in steps that are either one unit to the right or one unit up.

Example List all subsets of {a,b,c}.

6. A set with n distinct elements has exactly 2^n subsets.

Example The set {a,b,c,d,e} has $2^5 = 32$ subsets.

Example (a) How many subsets does {1,2,3,...,10} have?

(b) How many subsets of size 2 does {1,2,3,...,10} have?

7. The number of subsets or combinations of size r in a set with n distinct elements is given by

$$C(n,r) = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Example The number of subsets of {1,2,3,...,20} of size 11 is

$$\binom{20}{11} = \frac{20!}{11!9!} = 167{,}960.$$

Example A group of students consists of 8 seniors and 5 juniors. How many ways are to select a committee from these 13 students consisting of 2 juniors and 3 seniors?

Example How many bit stings of length 10 contain exactly 5 ones?

Solution Think of the positions of the bit strings as 1, 2, 3, ..., 10.

Given a bit string with 5 ones we can obtain a subset of size 5 of {1,2,...,10} from the locations of the 5 ones.
For example, 1001101010 corresponds to the subset {1,4,5,7,9}.

Thus there is a one-to-one correspondence between the bit strings of length 10 having 5 ones and subset of {1,2,...,10} size 5.

Therefore, there are $\binom{10}{5} = \frac{10!}{5!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2} = 252$ bit string of length 10 with 5 ones.

Example (a) Suppose that Steve draws 5 cards from a standard deck of 52 cards. How many different hands are possible?

(b) In how many ways can her selection result in a hand with no clubs?

(c) In how many ways can her selection result in a hand with at least one clubs?

(d) In how many ways can her selection result in a hand with exactly one club clubs?

Example (a) How many ternary strings are there of length 10?

(b) How many ternary strings are there of length 10 with exactly 4 ones?

(c) How many ternary strings are there of length 10 with an even number of ones?

8. The Binomial Theorem For any positive integer n:

$$(x+y)^n = \binom{n}{0}y^n + \binom{n}{1}xy^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots + \binom{n}{r}x^r y^{n-r} + \ldots + \binom{n}{n}x^n$$

Example Use the Binomial Theorem to expand $(x + 1)^5$.

$$(x+1)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$$

$$= 1 + 5x + 10x^2 + 15x^3 + 10x^4 + x^5$$

Example What is the coefficient of x^7 in the expansion of $(x + 1)^{10}$?

Example Use the Binomial Theorem to prove that the number of subsets in a set with n elements is 2^n.

Example Prove that the number of subsets with an even number of elements equals the number of subsets with an odd number of elements.

9. There are C(n + r – 1,r) r-combinations from a set with n elements when repetition of elements is allowed.

Example An ice cream parlor has 5 flavors. How many different ways can a dish of 3 scoops be made where each flavor can be used more than once and the order of the scoops does not matter?

Solution
Imagine a hypothetical situation with the following 5 flavors. Stars represent scoops of ice cream.

Vanilla   Chocolate   Cookies & Cream   Rocky Road   Chocolate Chip

*   /   /   /   **   /

Notice that the order can be thought of as an arrangement of 3 *'s and 4 /'s.

For example, one scoop of chocolate, one scoop of cookies and cream and one scoop of chocolate chip is the arrangement.

/ * / * / / *

What is the arrangement for 3 scoops of Rocky Road?

The number of ways to a dish of 3 scoops can be ordered is equal to the number of subsets of size 3 in a set with 7 elements $= \binom{7}{3} = 35$.

Example A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, how many different ways can we select a dozen donuts?

Example Determine all integer solutions to $x_1 + x_2 + x_3 + x_4 = 7$.

The Catalan Numbers

Example How many paths are there from (0,0) to (3,3) on an integer lattice that never go above the line $y = x$? Each edge must either go one unit up or one unit to the right.

1. For any integer $n \geq 1$, the number of paths (made up of n R's and n U's) going from (0,0) to (n,n) without rising above the line $y = x$ is given by

$$b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, \text{ where } b_0 = 1.$$

The numbers $b_n =$ are called the <u>Catalan numbers</u>.

Example (a) How many ways can one arrange three 1's and three -1's so that all six partial sums are nonnegative?
(b) How many ways can one arrange five 1's and five -1's so that all ten partial sums are nonnegative?

Logical Equivalence

1. Two statements s_1 and s_2 are called <u>logically equivalent</u>, denoted $s_1 \Leftrightarrow s_2$ if s_1 is true if and only if s_2 is true (i.e. s_1 and s_2 have the same truth table.)

Example Determine if the given pairs are logically equivalent.

(a) $\neg p \vee q$ and $p \rightarrow q$

(b) $\neg(p \vee q)$ and $(\neg p \vee \neg q)$

Example Use truth tables to prove DeMorgan's Law $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$.

2. A <u>tautology</u> is a statement that is always true and is denoted by T_0.
A <u>contradiction</u> is a statement that is always false and is denoted by F_0.

3. Let s be a statement that contains no logical connectives other than $\wedge$ and $\vee$. Then the <u>dual</u> of s, denoted by s^d, is the statement obtained from s by replacing:

$$\wedge \text{ with } \vee,\ \ \vee \text{ with } \wedge,\ T_0 \text{ with } F_0, \text{ and } F_0 \text{ with } T_0.$$

If p is a primitive statement (i.e. does not contain connectives), then p^d is the same as p, and $(\neg p)^d$ is the same as $\neg p$.

Example If s is given by $(p \wedge \neg q) \vee (r \wedge T_0)$ where p,q and r are primitive statements, then s^d is given by $(p \vee \neg q) \wedge (r \vee F_0)$.

4. Duality Principle: If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

Example In a previous example we proved DeMorgan's Law $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$.

The Duality Principle implies that $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$.

5. Using the concepts of logical equivalence, and duality we can prove a list of laws of logic.

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6. Laws of Logic can be used to simplify statements, which are also used in circuit theory.

Example Simplify $(p \vee q) \wedge \neg\, (\neg p \wedge q)$.

	Steps	Reasons
	$(p \vee q) \wedge \neg\, (\neg p \wedge q)$	Given
$\Leftrightarrow$	$(p \vee q) \wedge (\neg\, \neg p \vee \neg q)$	DeMorgan's Law
$\Leftrightarrow$	$(p \vee q) \wedge (p \vee \neg q)$	Double Negation
$\Leftrightarrow$	$p \vee (q \wedge \neg q)$	Distributive Law
$\Leftrightarrow$	$p \vee F_0$	Inverse Law
$\Leftrightarrow$	p	Identity Law

7. Switching circuits can be represented by logic statements and then simplified using laws of logic.

Example Find a logic statement that represents the given circuit.

Example Draw a circuit that corresponds to $[(p \wedge \neg q) \vee (r \vee q)] \wedge s$

Example Find a logic statement to represent each circuit, then simplify the statement and draw the corresponding simplified circuit.

Logical Implication and Rules of Inference

1. A logical argument is a statement of the form $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$.

$p_1, p_2, \ldots, p_n$ are called premises and q is called conclusion.

2. An argument is called valid if whenever all of the premises are true, the conclusion is also true.

Example Let p, q and r denote the primitive statements:

p: Roger studies
q: Roger plays racketball
r: Roger passes

Let p_1, p_2, and p_3 denote the premises:

p_1: If Roger studies, then he will pass.
p_2: If Roger does not play racketball, then he will study.
p_3: Roger failed.

Notice that $p_1 : p \to r$ $\quad p_2 : \neg q \to p$ $\quad p_3 : \neg r$

Thus the argument can be analyzed by considering

$$[(p \to r) \wedge (\neg q \to p) \wedge (\neg r)] \to q$$

The argument is valid if whenever p_1, p_2, and p_3 are all true, q is also true.

One way is to use a truth table and show that the argument is a tautology.

Example Construct a truth table for $[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge (\neg r)] \rightarrow q$ and determine if the argument is valid.

3. Certain arguments occur often and are called <u>rules of inference</u>. For example consider the following argument called Modus Tollens.

$$\begin{array}{l} p \rightarrow r \\ \underline{\neg q \quad\quad} \\ \therefore \neg p \end{array}$$

Show that this is valid.

4. There is a list of well known rules of inference. These rules together with laws of logic can be used can be used to show that arguments are valid.

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Example Establish the validity of the argument

$p \to q$
$q \to (r \wedge s)$
$\neg r \vee (\neg t \vee u)$
$\underline{p \wedge t \qquad\qquad\qquad}$
$\therefore u$

Steps	Reasons
1. $p \to q$	Premise
2. $q \to (r \wedge s)$	Premise
3. $p \to (r \wedge s)$	1 & 2 & Syllogism
4. $p \wedge t$	Premise
5. p	4 & Conjunctive Simplification
6. $r \wedge s$	5 & 3 & Detachment
7. r	6 & Conjunctive Simplification
8. $\neg r \vee (\neg t \vee u)$	Premise
9. $\neg(r \wedge t) \vee u$	8 & Associative Law of $\vee$ & DeMorgan's
10. t	4 & Conjunctive Simplification
11. $r \wedge t$	7 & 10
12. u	9 & 11 & Double Negation & Disjunctive Syllogism

Example Show that the following argument is valid.

If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were cancelled, then the refunds would have had to be made. No refunds were made.

Therefore the band could play rock music.

Solution First we define the following.

p: The band could play rock music
q: The refreshments were delivered on time
r: The New Year's party was canceled.
s: Alicia was angry
t: Refunds had to be made

The argument is given by:

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline \therefore p \end{array}$$

The validity is established as follows.

Steps	Reasons
1. $r \rightarrow t$	Premise
2. $\neg t$	Premise
3. $\neg r$	(1) & (2) &Modus Tollens
4. $\neg r \vee \neg s$	(3) & Disjunctive Amplification
5. $\neg (r \wedge s)$	DeMorgan's
6. $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
7. $\neg (\neg p \vee \neg q)$	(5) & (6) & Modus Tollens
8. $(p \wedge q)$	DeMorgan's and Double Negation
9. $\therefore p$	(8) & Conjunctive Simplification

The Use of Quantifiers

1. A <u>propositional function</u> or an <u>open statement</u> p(x) is a statement which contains the variable x from a designated set called the universe of discourse and denoted by U.

x is called the variable

The universe of discourse U can be any set. Typically it is either
Z = integers or R = reals

Example Let U = Z, and let p(x) be given by: $x > 0$. Determine the truth value of the following.

p(3) (true)   p(0) (false)

p(-3)   p(.5)

Example Let r(x,y) denote $x = y + 3$. Where x and y are reals.
What are the truth values of:

(a) r(0,3)

(b) r(3,0)

2. The <u>universal quantification</u> of p(x) is the proposition

"P(x) is true for all values of x in U" denoted by $\forall x\, p(x)$.

Example Let U be the set of all reals, and let p(x) denote $x^2 = 1$, then is $\forall x\ p(x)$ true?

Example Let p(x) be given by: $x^2 \geq 1$. Determine the truth value of $\forall x\ p(x)$ with the following U.

(a) U = positive integers

(b) U = all integers

(c) U = all reals

Example Determine if $\forall n\ p(n)$ is true for the following p(n). Assume that U is the set of integers.

(a) p(n) : $n^2 > 0$

(b) p(n) : n(n + 1) is even

Example If all of the elements of U can be listed as $x_1, x_2, \ldots x_m$, then $\forall x\ p(x)$ is logically equivalent to $p(x_1) \wedge p(x_2) \wedge \ldots \wedge p(x_m)$

3. The <u>existential quantification</u> of p(x) is the proposition "There exists an element x in U such that p(x) is true" denoted by $\exists x\ p(x)$.

Example Let p(x) be given by: $x^2 < 1$.
Determine the truth value of $\exists x\ p(x)$ with the following U.

(a) U = positive integers

(b) U = all integers

(c) U = all reals

Example Express the statement "The integer 41 is equal to the sum of two perfect squares using quantifiers.

Let U be the set of positive integers, then the statement is

$$\exists m\ \exists n\ [41 = m^2 + n^2]$$

Example Determine if $\exists x\ P(x)$ is true for the following P(x). Assume that U = R.

(a) P(x): $x^2 > 0$

(b) P(x): $x + 1 < x/2$

(c) P(x): $x^2 \leq 0$

Example If all of the elements of U can be listed as $x_1, x_2, \ldots x_m$, then $\exists x\ p(x)$ is logically equivalent to $p(x_1) \vee p(x_2) \vee \ldots \vee p(x_m)$

Example Let P(x) denote "$x^2 = 5$" where U is the set of positive reals. Determine the truth value of the following.

(a) P(2) (b) $\exists\ x\ P(x)$ (c) $\forall\ x\ P(x)$

4. The following logical equivalences involving quantifiers hold.

$$\neg[\forall x\ P(x)] \Leftrightarrow \exists x\ \neg P(x)$$

$$\neg [\exists x\ P(x)] \Leftrightarrow \forall x\ \neg P(x)$$

Note: Distributing a negation operator across a quantifier changes a universal quantifier to an existential and vice versa.

Example Let U be the set of all students in Math 6617. Let P(x) denote " x has studied Java". Express the given statement using an English sentence. Also, determine which are logically equivalent?

(a) $\forall x\ P(x)$

(b) $\exists x\ P(x)$

(c) $\neg\forall x\ P(x)$

(d) $\exists x\ \neg P(x)$

(e) $\neg\exists x\ P(x)$

(f) $\forall x\ \neg P(x)$

Nested Quantifiers

Multiple quantifiers must be read from left to right.

Example Assume that the universe of discourse for x and y is R.

Let Q(x,y) denote : $x + y = 17$.

$\forall x \exists y\, Q(x,y)$ is the statement that for every real number x there is a real number y such that $x + y = 17$.

For example, if $x = 3$, then $y =$

if $x = 5$, then $y =$

if $x = -10.67$, then $y =$

Notice how y depends on x.

$\exists x \forall y\, Q(x,y)$ is the statement that there exists a real number x such that for every real number y, $x + y = 17$.

Here the same x must hold for all y. This is false because if $y = 3$, then x must be 14, but if $y = 5$, then x must be 12. So the same value for x does not work for all y.

Example Let Q(x,y) denote "y is the father of x" where x and y vary over the set of all males. Translate each of the following into an English sentence its determine truth value.

(a) $\forall x \exists y\, Q(x,y)$

(b) $\exists y \forall x\, Q(x,y)$

(c) $\exists x \forall y\, Q(x,y)$

Example Let U = {1,2,3}. Find an expression equivalent to $\forall x \exists y\, Q(x,y)$ using $\wedge$ and $\vee$.

Example The following are logically equivalent.

(a) $\neg [\forall x \exists y\, Q(x,y)] \Leftrightarrow \exists x \forall y\, \neg Q(x,y)$

(b) $\neg [\exists x \forall y\, Q(x,y)] \Leftrightarrow \forall x \exists y\, \neg Q(x,y)$

Example Express "Everyone has exactly one best friend" as a logical expression.

Solution. Let B(x,y) denote "y is the best friend of x".

The given statement says that for every person x there is another person y such that y is the best friend of x, and if z is some person other than y, then z is not the best friend of x.

$$\forall x \exists y \forall z\, [B(x,y) \wedge (z \neq y \rightarrow \neg B(x,z))]$$

Example (a) Give the definition of $\lim_{x\to a} f(x) = L$.
(b) Express the definition using quantifiers.

Solution
(a) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that,

$0 < |x - a| < \delta$ implies $0 < |f(x) - L| < \varepsilon$.

(b) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x\, [|x - a| < \delta \rightarrow 0 < |f(x) - L| < \varepsilon]$

where the universe of discourse is the set of reals which is restricted by the statements.

Sets and Subsets

1. A set can be defined by either listing elements or using set builder notation.

$A = \{1,2,3,4,5\}$ $A = \{x: x \text{ is an integer and } 1 \leq x \leq 5\}$

2. Set Notation

U is used to denote a universal set.

$\varnothing$ is used to denote the empty set.

If a is an element of A, we write $a \in A$.

$A \subseteq B$ means A is a subset of B.

$A \subset B$ means A is a subset of B, and $A \neq B$.

$|A|$ is used to denote the number of elements in A which is called the

<u>cardinality</u> of A.

$A = B$ if $A \subseteq B$ and $B \subseteq A$.

Example Let $U = \{1,2,3, \ldots ,30\}$. Let $A = \{10, 20, 30\}$ and $B = \{5,10,20,25,30\}$.

(a) What is $|A|$?
(b) What is $|B|$?
(c) Is $A \subseteq B$? Is $A \subset B$?
(d) Is $B \subseteq A$?
(e) Is $5 \in A$? Is $5 \in B$?

3. If A is a set from universe U, the <u>power</u> set of A, denoted P(A) is the set of all subsets of A. If A has n elements then P(A) has 2^n elements.

Example How many ways can one write 7 as the sum of one or more positive integers where the order of summands is relevant? (i.e. how compositions of 7 are there?

We will show that the answer can be obtained by establishing a one-to-one correspondence between compositions of 7 and subsets of {1,2,3,4,5,6}.

Consider the plus signs in the following sum.

1	+	1	+	1	+	1	+	1	+	1	+	1
	1st		2nd		3rd		4th		5th		6th	

Given a subset we carry out the corresponding addition.

For example the subset {1,2,5} would give the composition 3 + 1 + 2 + 1.

Given a composition we determine which additions were carried out to obtain a subset of {1,2,3,4,5,6}.

For example, the composition 2 + 3 + 1 + 1 would yield the subset {1,3,4}.

Find the composition corresponding to {2,3,4,6}.

Find the subset corresponding to 1 + 4 + 2.

Since the number of subsets is 2^6, there are 64 compositions.

Example List all the compositions of 4 and give the corresponding subsets.

4. Pascal's Identity $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

Example Verify that $\binom{6}{3} + \binom{6}{4} = \binom{7}{4}$.

Example Prove Pascal's Identity.

Example (a) Construct Pascal's Triangle up to n = 6.
(b) Expand $(x + 1)^6$.

5. Set of numbers

Z = integers = $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$

N = natural numbers = $\{ 0, 1, 2, 3, \ldots \}$

Z^+ = positive integers = $\{1, 2, 3, \ldots \}$

Q = rationals = $\{ a/b : a, b \in Z, b \neq 0\}$

Q^+ = positive rationals = $\{ r \in Q : r > 0 \}$

Q^* = nonzero rationals

R = reals

Set Operations

1. A union B, denoted $A \cup B$, is the set of elements in A or B

   A intersection B, denoted $A \cap B$, is the set of elements in A and B

   The symmetric difference of A and B, denoted $A \Delta B$ is the set of elements in A or B, but not in both.

Example Let A = {1,3,4,5,6,8,9} and B = {2,4,7,8,10}. Find $A \cup B$, $A \cap B$, and $A \Delta B$.

2. A and B are <u>disjoint</u> if $A \cap B = \emptyset$.

3. Let U be a universal set containing subset A. Then the <u>complement</u> of A is the set of elements in U but not in A denoted by either $U - A$, or $\overline{A}$.

4. Prove DeMorgan's Law $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using set notaton.

Let $x \in \overline{A \cup B}$ $\Rightarrow$ $x \notin A \cup B$
$\Rightarrow$ $x \notin A$ and $x \notin B$
$\Rightarrow$ $x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow$ $x \in \overline{A} \cap \overline{B}$
This implies that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Let $x \in \overline{A} \cap \overline{B}$ $\Rightarrow$ $x \in \overline{A}$ and $x \in \overline{B}$
$\Rightarrow$ $x \notin A$ and $x \notin B$
$\Rightarrow$ $x \notin A \cup B$
$\Rightarrow$ $x \in \overline{A \cup B}$
This implies that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

5. Use a membership table to prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. Proofs of set identities can also be constructed using the Laws of Set Theory.

Example Show that $[(A \cup B) \cap C] \cap B = B \cap C$

$[(A \cup B) \cap C] \cap B = (A \cup B) \cap [C \cap B]$ Associative Law of Int.

$= (A \cup B) \cap [B \cap C]$ Comm. Law of Int.

$= [(A \cup B) \cap B] \cap C$ Associative Law of Int.

$= B \cap C$ Absorption Law

Example Show that $\overline{(A \cup B)} \cup (A \cap B) = A \,\Delta\, \overline{B}$

Counting and Venn Diagrams

<u>Inclusion-Exclusion Principle</u> Given a set A, let $|A|$ denote the number of elements in A. Then $|A \cup B| = |A| + |B| - |A \cap B|$

Example How many integers are there between 1 and 30 that are multiples of 3 or 5?

Solution Let A = {3,6,9,12,15,18,21,24,27,30} and B = {5,10,15,20,25,30}. We must find $|A \cup B|$.

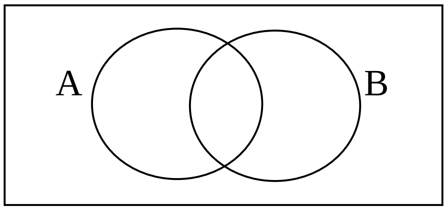
Since $A \cap B = \{15,30\}$, the inclusion-exclusion principle gives

$$|A \cup B| = |A| + |B| - |A \cap B| = 10 + 6 - 2 = 14.$$

Alternative Solution. List the elements in $A \cup B$ and count them.

$$A \cup B = \{3,5,6,9,10,12,15,18,20,21,24,25,27,30\}.$$

Venn Diagram Solution. Place the elements in the appropriate region.



Example How many integers are there between 1 and 100 that are multiples of 5 or 6?

Example How many bit strings of length 5 either start with 1 or end with 1?

Example How many bit strings of length 8 either start with 1 or end with 00?

Discrete Probability

1. An experiment is a procedure that yields one of a given set of outcomes.

2. The sample space of an experiment is the set of all possible outcomes of an experiment.

3. An event is a subset of the sample space.

4. The probability of an event E which is a subset of a sample space S with equally likely outcomes is

$$p(E) = \frac{|E|}{|S|}.$$

Example A computer simulation involves the random generation of digits from 1 to 5. If two digits are generated, find the probability that they are both odd if:
(a) The first digit cannot be generated again.
(b) The first digit can be generated again.

Example A permutation of {1,2,3,4,5,6} is selected at random.
(a) What is the probability that it begins with a 5 or 6?
(b) What is the probability that 2 is followed immediately by 1?
(c)What is the probability that it has the form odd, even,odd,even,odd,even?

Example What is the probability that a positive integers selected at random from {1,2,3,..., 100} is divisible by either 2 or 5?

Example What is the probability of winning a lottery in which a person must correctly choose 6 numbers out of 40? Assume that no numbers can be repeated.

Solution The total number of combinations is C(40,6) = 3,838,380. So the probability is 1/3,838,380 = .00000026.

Example What is the probability that a random 4-digit telephone extension has one or more repeated digits?

Solution The total number of 4-digit extensions is 10^4 = 10,000.

Prob. of 1 or more repeated digits = 1 – Prob. of no repeated digits.

Prob. of no repeated digits = $\frac{10 \cdot 9 \cdot 8 \cdot 7}{10^4}$ = .504.

Prob. of 1 or more repeated digits = 1 – .504 = .496.

Example Find the probability that a random arrangement of the letters WYSIWYG

(a) starts and ends with a W;
(b) starts and ends with the same letter;

Example The Freshman class at a college has 300 CS students. 180 can program in Java, 120 in VB, 30 in C++, 12 in Java and C++, 18 in VB and C++, 12 in Java and VB and 6 in all 3.

(a) If a student is selected at random, what is the probability that she can program in exactly two languages?

(b) If a student is selected at random, what is the probability that she can program in none of the 3 languages?

(c) If two students are selected at random, what is the probability that they can both program in Java?

The Well Ordering Principle and Mathematical Induction

1. <u>The Well Ordering Principle</u>: Every nonempty set of positive integers contains a smallest element.

2. <u>The Principle of Mathematical Induction</u> Let S(n) be an open mathematical statement with universe of discourse Z^+. A proof by <u>mathematical induction</u> that S(n) is true for every positive integer n consists of two steps:

(i) *Basis Step* Show that S(1) is true.

(ii) *Inductive Step* Show that if S(k) is true for some positive integer k, then S(k + 1) is true.

Example Use induction to prove that $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Let S(n) be the statement: $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Then S(1) asserts that $1 = 1^2$, which is true. This proves the basis step.

To establish the inductive step we must show that $S(k) \rightarrow S(k + 1)$.

So we assume that $1 + 3 + 5 + ... + (2k - 1) = k^2$, for some $k \geq 1$.

We must show that $1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = (k + 1)^2$.

Start with the left side of what we must show.

$[1 + 3 + 5 + ... + (2k - 1)] + (2k + 1) = k^2 + (2k + 1)$ This follows from the ind. assumption.

$[1 + 3 + 5 + ... + (2k - 1)] + (2k + 1) = (k + 1)^2$ Factor

We are done!

Example Let f(n) be the function from $\mathbf{Z}^+$ to $\mathbf{R}$ given by

$$f(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}.$$

(a) Calculate f(2), f(3) and f(4).
(b) Conjecture a formula for f(n).
(c) Use induction to prove that your formula is correct.
(d) Calculate f(50).

Example Let f(n) be the function from $\mathbf{Z}^+$ to **R** given by

$$f(n) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2n-1)(2n+1)}.$$

(a) Calculate f(2), f(3) and f(4).
(b) Conjecture a formula for f(n).
(c) Use induction to prove that your formula is correct.
(d) Calculate f(50).

Example Prove or disprove: If f is a function from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ given by f(n) = $n^2 + n + 17$, then f(n) is prime.

3. Let S(n) be an open mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer. Let n_0 and n_1 be positive integers with $n_0 \leq n_1$. A proof by mathematical induction that S(n) is true for every integer $n \geq n_0$, consists of two steps:

(i) *Basis Step* Show that $S(n_0)$, $S(n_0+1)$, ... $S(n_1)$ is true.

(ii) *Inductive Step* Show that if $S(n_0)$, $S(n_0+1)$, ... S(k) is true, then S(k+1) is true.

Example Prove that for every integer n ≥ 14, n can be expresses as a sum of 3's and/or 8's.

We begin by showing that S(14), S(15) and S(16) are true.

14 = 3 + 3 + 8
15 = 3 + 3 + 3 + 3 + 3
16 = 8 + 8

For the inductive step suppose that S(14), S(15), S(16), ..., S(k) are true where k ≥ 16. We must show that S(k+1) is true.

First note that $k + 1 = 3 + (k - 2)$.

By the inductive assumption k – 2 is the sum of 3's and 8's. Since k + 1 is 3 plus a sum of 3's and 8's S(k+1) is true.

Example Use the above proof to express 17, 18, 19 and 20 as a sum of 3's and 8's.

Example Determine which values of n satisfy $2^n < n!$ and prove your result using induction.

n	2^n	n!
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120

We will show that for every $n \geq 4$, we have $2^n < n!$

Let S(k) be the statement: $2^k < k!$

Then S(4) asserts that $2^4 < 4!$, which was computed above and is true. This proves the basis step.

To establish the inductive step we must show that $S(k) \rightarrow S(k + 1)$.

So we assume that $2^k < k!$, for some $k \geq 4$.

We must show that $2^{k+1} < (k+1)!$.

We will start with the left side and end with the right side.

$$\begin{aligned}
2^{k+1} &= 2\cdot 2^k && b^m b^n = b^{m+n} \\
&< 2 \cdot k! && \text{By the inductive assumption } 2^k < k! \\
&< (k+1) \cdot k! && \text{Since } k \geq 4,\quad 2 < k + 1 \\
&= (k+1)! && (k+1)! = (k+1) \cdot k!
\end{aligned}$$

Example Determine which values of n satisfy $n! < n^n$ and prove your result using induction.

Recursive Functions and Sets

1. Let f be a function from $\mathbf{Z}^+ \cup \{0\}$ to R. f(n) is called <u>recursively defined</u> if:

(i) f(0) is specified
(ii) f(n+1) is determined from f(k) where $k \leq n$.

Example Let $f(0) = 3$ and $f(n) = 2f(n-1) + 10$. Find f(4).

Solution

$$\begin{aligned} f(1) &= 2f(0) + 10 = 2(3) + 10 &&= 16 \\ f(2) &= 2f(1) + 10 = 2(16) + 10 &&= 42 \\ f(3) &= 2f(2) + 10 = 2(42) + 10 &&= 94 \\ f(4) &= 2f(3) + 10 = 2(94) + 10 &&= 198 \end{aligned}$$

Example Let $f(0) = 3$ and $f(n+1) = n \cdot f(n)$.

(a) Find f(5).
(b) What function is f?

Example Let $f(0) = 5$ and $f(n+1) = 2f(n)$.

(a) Find f(3)
(b) Give an explicit formula for f(n).
(c) Verify that your formula is correct using induction.

2. The <u>Fibonacci numbers</u> $f_0, f_1, \ldots$ are defined by:

$$f_0 = 0,\ f_1 = 1, \text{ and } f_{n+1} = f_n + f_{n-1}.$$

Example Find f_6.

Solution

$$\begin{aligned} f_2 &= f_1 + f_0 = 1 + 0 = 1 \\ f_3 &= f_2 + f_1 = 1 + 1 = 2 \\ f_4 &= f_3 + f_2 = 2 + 1 = 3 \\ f_5 &= f_4 + f_3 = 3 + 2 = 5 \\ f_6 &= f_5 + f_4 = 5 + 3 = 8 \end{aligned}$$

Example Prove that $(f_0)^2 + (f_1)^2 + ... + (f_n)^2 = f_n f_{n+1}$, for every $n \geq 1$.

Before proving this lets see if it is plausible.

n	$(f_0)^2 + (f_1)^2 + ... + (f_n)^2$	$f_n f_{n+1}$
1	$(f_0)^2 + (f_1)^2 = 0^2 + 1^2 = 1$	$f_1 f_2 = 1 \cdot 1 = 1$
2	$(f_0)^2 + (f_1)^2 + (f_2)^2 = 0^2 + 1^2 + 1^2 = 2$	$f_2 f_3 = 1 \cdot 2 = 2$
3	$(f_0)^2 + (f_1)^2 + (f_2)^2 + (f_3)^2 = 0^2 + 1^2 + 1^2 + 2^2 = 6$	$f_3 f_4 = 2 \cdot 3 = 6$
4	?	
5	?	

The proof is by induction. The case n = 1 is given above.

Inductive Assumption: $(f_0)^2 + (f_1)^2 + ... + (f_n)^2 = f_n f_{n+1}$, for some $n \geq 1$.

We must show that $(f_0)^2 + (f_1)^2 + ... + (f_n)^2 + (f_{n+1})^2 = f_{n+1} f_{n+2}$

$(f_0)^2 + (f_1)^2 + ... + (f_n)^2 + (f_{n+1})^2$	$= f_n f_{n+1} + (f_{n+1})^2$	Inductive Assumption
	$= f_{n+1}(f_n + f_{n+1})$	Factor
	$= f_{n+1} f_{n+2}$	$f_{n+2} = f_n + f_{n+1}$, by definition of the Fibonacci numbers

Example A person invests \$5,000 into an account that pays 8% interest compounded annually. If the account is left untouched for 15 years, how much will the account be worth at the end of 15 years?

Solution Let $f(n)$ be the amount that the account is worth after n years. Then $f(0) = 5{,}000$ and we want to find $f(15)$.

Notice that f satisfies $f(n+1) = 1.08\, f(n)$. This leads to an explicit formula for f given by $f(n) = 5{,}000(1.08)^n$.

This gives $f(15) = 5{,}000(1.08)^{15} = 5{,}000(3.172) = 15{,}860$.

Example If $f(0) = 1$, $f(1) = 1$, $f(2) = 1$, and $f(n+1) = f(n) + f(n-1) + f(n-2)$.

What is $f(7)$?

Example Let $f(0) = 2$ and $f(n+1) = f(n) + 2(n+1)$.

Prove that $f(n) = n^2 + n + 2$ for all $n \geq 0$.

Solution The proof is by induction.

When $n = 0$, we have $f(0) = 0 + 0 + 2 = 2$. This proves the Basis Step.

Inductive Assumption: $f(n) = n^2 + n + 2$ for some $n \geq 0$.

We must show that $f(n+1) = (n+1)^2 + (n+1) + 2$.

$$\begin{aligned}
f(n+1) &= f(n) + 2(n+1) && \text{By definition of } f \\
&= (n^2 + n + 2) + 2(n+1) && \text{By the inductive assumption} \\
&= n^2 + 3n + 4 && \text{Simplify} \\
&= (n^2 + 2n + 1) + (n+1) + 2 && \text{Regroup} \\
&= (n+1)^2 + (n+1) + 2. && \text{Factor}
\end{aligned}$$

3. A set is called recursively defined if:

(i) An initial collection of elements is given; and
(ii) Rules are given to construct new elements of the set using other elements already known to be in the set.

Example What is the set defined by

$3 \in S$
if $x \in S$ and $y \in S$, then $x + y \in S$?

Solution We are given $3 \in S$.

Taking $x = 3$ and $y = 3$ we get that $3 + 3 = 6 \in S$.

Taking $x = 3$ and $y = 6$ we get that $3 + 6 = 9 \in S$.

Taking $x = 3$ and $y = 9$ we get that $3 + 9 = 12 \in S$.

Conclusion: S consists of all positive multiples of 3.

Example Give a recursive definition of the odd positive integers.

Solution $1 \in S$
if $x \in S$, then $x + 2 \in S$

Example Give a recursive definition of the set S of positive powers of 5 ($S = \{5, 25, 125, ...\}$.)

The Division Algorithm and Primes

1. If a and b are integers, $a \neq 0$, we say that a <u>divides</u> b if there is an integer c such that $b = ac$, this is denoted by $a \mid b$.

Example 5 divides 20 since $20 = (4)(5)$
5 does not divide 21

If a divides b, then b is called a <u>multiple</u> of a.

An integer b is <u>even</u> if 2 divides b, otherwise it is <u>odd</u>.

2. A positive integer $p > 1$ is <u>prime</u> if the only positive integers that divide p are 1 and p. If p is not prime, then it is called <u>composite</u>.

Example Which of the following are prime?

1 2 8 9 19 2003

Example Determine if the given number is prime or composite. If it is composite find a prime divisor.

(a) 57

(b) 119

(c) 22,257

3. **Theorem** Every integer $n > 1$ is either prime or has a prime divisor.

4. **Theorem** There are infinitely many primes.

Proof. Suppose to obtain a contradiction there is a finite number of primes.

Let p_1, ..., p_k, denote this set of primes in increasing order.

Let $B = (p_1)(p_2) ...(p_k) + 1$.

Then B is not prime since $B > (p_k)$. So B must be composite.

This implies that B must have a prime divisor. Therefore, some p_i must divide B.

But when B is divided by any prime p_i, the remainder is 1. This is a contradiction.

5. The Division Algorithm Let d be a positive integer and a an integer. Then there are unique integers q and r with $0 \leq r < d$ such that $a = dq + r$.

Example Complete the following table

a	d	q	r	Relationship
32	4	8	0	32 = (4)(8)
55	13			
–35	4			
–26	3			

Example Use the Division Algorithm to write 6137 in base 8.

Solution We first must express 6137 as $r_0 + r_1 \cdot 8 + r_2 \cdot 8^2 + ...+ r_k \cdot 8^k$.

r_0 is the remainder when 6137 is divided by 8, so 8137 = 1 + 8·767, implies $r_0 = 1$.

r_1 is the remainder when 767 is divided by 8, so 767 = 7 + 8·95, gives $r_1 = 7$.

r_2 is the remainder when 95 is divided by 8, 95 = 7 + 8·11, gives $r_2 = 7$.

r_3 is the remainder when 11 is divided by 8, 11 = 3 + 8·1, gives $r_3 = 3$.

r_4 is the remainder when 1 is divided by 8, 1 = 1 + 8·0, gives $r_4 = 1$.

This gives $6137 = 1 + 7 \cdot 8 + 7 \cdot 8^2 + 3 \cdot 8^3 + 1 \cdot 8^4$.

Check $1 + 7 \cdot 8 + 7 \cdot 8^2 + 3 \cdot 8^3 + 1 \cdot 8^4 = 1 + 56 + 448 + 1536 + 4096 = 6137$

$6137 = (13771)_8$.

Example Use the Division Algorithm to write 100 in base 2.

In base 16 we run out of numeric symbols, so we use letters as follows.

Base 10	10	11	12	13	14	15
Base 16	A	B	C	D	E	F

Example Convert 4C2 to base 10 and then to base 2.

$4C2 = 2 + C \cdot 16 + 4 \cdot (16)^2 = 2 + 12 \cdot 16 + 4 \cdot 256 = 2 + 192 + 1024 = 1218.$

$1218 = 2^{10} + 2^7 + 2^6 + 2 = (10011000010)_2$

Example Use the Division Algorithm to write 1234 in base 16.

6. Theorem If n is composite, then n has a prime divisor less than or equal to $\sqrt{n}$.

Proof Assume that n is composite.

By def. of composite, n has a factor, say a, with $1 < a < n$.

Since a divides n, there is an integer b such that $n = ab$, where an and b are both positive integers greater than 1.

If both $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab > \sqrt{n}\sqrt{n} = n$, which is impossible. Therefore, either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Consequently, n has a positive divisor that is at most $\sqrt{n}$.

This divisor is either prime (in which case we are done) or by an earlier theorem, it has a prime divisor, which must also divide n. In either case, n has a prime divisor that is at most $\sqrt{n}$.

Example Given the following values for n, find a prime divisor that is at most $\sqrt{n}$ if possible.

(a) 20 (b) 75 (c) 37 (d) 2047

The Greatest Common Divisor and The Euclidean Algorithm

1. Let a, b $\in$ Z, where a $\neq$ 0 and b $\neq$ 0. Then c $\in Z^+$ is called a <u>greatest common divisor</u> of a and b

(i) if c divides a, and c divides b;

(ii) for any common divisor d of a and b, d divides c

The notation d = gcd(a,b) is often used.

Example gcd(10,25) = 5

gcd(14,35) = ?

2. The Euclidean Algorithm

Let a, b $\in Z^+$ with a > b. Set r_0 = a and r_1 = b and apply the division algorithm n times as follows.

$r_0 = q_1r_1 + r_2 \qquad 0 < r_2 < r_1$

$r_1 = q_2r_2 + r_3 \qquad 0 < r_3 < r_2$

$r_2 = q_3r_3 + r_4 \qquad 0 < r_4 < r_3$

.
.
.
$r_{n-2} = q_{n-1}r_{n-1} + r_n \qquad 0 < r_n < r_{n-1}$

$r_n = q_nr_n$

Then r_n, the last nonzero remainder is the gcd(a,b).

Example Find the gcd(414,662) and express the result as a linear combination of these integers.

$662 = 414\,(1) + 248$

$414 = 248(1) + 166$

$248 = 166(1) + 82$

$166 = 82(2) + 2$

$82 = 2\,(41)$

$\gcd(414,662) = 2$

The linear combination is obtained by working backwards.

$$\begin{aligned}
2 &= 166 - 82(2) \\
&= 166 - (2)(248 - 166) \quad = (3)166 - (2)248 \\
&= (3)(414 - 248) - (2)248 \quad = (3)414 - (5)248 \\
&= (3)414 - (5)(662 - 414) = (8)414 - (5)662
\end{aligned}$$

check $(8)414 - (5)662 = 3312 - 3310$

3. Let a be an integer and m a positive integer. Then a mod m is the remainder given by the Division Algorithm when a is divided by m.

Example

12 mod 5 = 2

13 mod 3 = ?

–12 mod 5 = ?

–13 mod 5 = ?

4. The least common multiple of a and b is denoted by lcm(a,b) and satisfies

$$ab = \gcd(a,b)\operatorname{lcm}(a,b)$$

Example lcm(414,662) = (414)(662)/gcd(414,662) = 137,034

Fundamental Theorem of Arithmetic

1.**Theorem** Every positive integer can be expressed uniquely as the product of primes, where the prime factors are given in increasing order.

Example	Integer	Prime Factorization
	100	$2^2 5^2$
	641	641
	999	
	1024	
	10!	

Example Use a tree to determine the prime factorization of 980,220

Example Show how to find the gcd and lcm using prime factorizations.

Example Prove that $\sqrt{2}$ is irrational.

Restatement: if r is a real number and $r^2 = 2$, then r is irrational.

Suppose that, to obtain a contradiction, r is a real number with $r^2 = 2$ and r is rational.

By definition of rational, there exists integers a and b $\neq 0$, such that r = $\frac{a}{b}$.

By factoring a and b into their prime factorizations and dividing out all common factors, we can find integers c and d $\neq 0$, such that r = $\frac{c}{d}$ and c and d have no common factors.

Substituting r = $\frac{c}{d}$ into $r^2 = 2$ gives $\frac{c^2}{d^2} = 2$.

Multiplying by d^2 gives $c^2 = 2d^2$.

This implies that c^2 is even, and by a prior example, we know that c must be even.

Since c is even there exists an integer m such that c = 2m.

Substituting c = 2m into $c^2 = 2d^2$ gives $(2m)^2 = 2d^2$, and then dividing by 2 gives $2m^2 = d^2$.

This implies that d^2 is even, and by a prior example, we know that d must be even.

Now we have shown that both c and d are even, which means they have a common factor of 2.

But this contradicts the fact that c and d have no common factors.

Cartesian Products and Relations

1. For sets A and B the Cartesian product of A and B, denoted by A × B is $\{(a,b): a \in A \text{ and } b \in B\}$

Example Let A = {1,3,5} and B = {2,4}. Find A × B.

Example What is R × R?

Example Let A = {0,1,2} Graph the set of points A × A.

2. For sets A and B any subset of A × B is called a <u>binary relation from A to B</u>. Any subset of A × A is called a <u>binary relation on A</u>.

Example Let A = {1,3,5} and B = {2,4}. The following are relations from A to B.

(a) {(1,2), (3,4)}
(b) {(1,2), (3,2), (5,2)}

Example Let A = {1,3,5} and B = {2,4}. How many different relation relations from A to B are there?

Example Let A = {1,3,5}

(a) List the elements in A × A.

(b) Is {(1,1), (3,5), (5,3)} a binary relation on A?

(c) Is {(1,1), (1,3), (1,5)} a binary relation on A?

Example Let A = {2,3,4}, B = {4,5} and C = {x,y}. Use a tree diagram to represent the following.

(a) A × B

(b) B × A

(c) A × B × C

3. For finite sets A and B with m elements in A and n elements in B, there are 2^{mn} relations from A to B including the empty relation and A × B.

Proof ?

Example Let R be the subset of N × N where R = {(m,n) : n = 7m}.

(a) List 3 elements of R.

(b) Give a recursive definition of R.

Functions

1. Let A and B be sets. A <u>function</u> f from A to B is an assignment of elements in B to elements in A such that every element in A is assigned exactly one element in B.

f(a) = b means that f assigns b to a

b is called the <u>image</u> of a under f

a is called the <u>preimage</u> of b under f

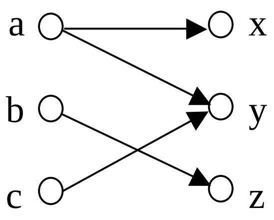
A is called the <u>domain</u>.

B is called the <u>codomain</u>.

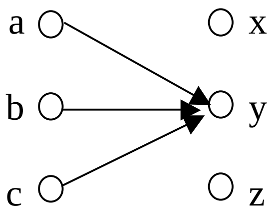
The <u>range</u> of A under f is the set of all images of points in A under f, denoted by f(A).

Example Which of the following are functions?

1.



2.



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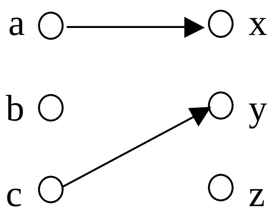

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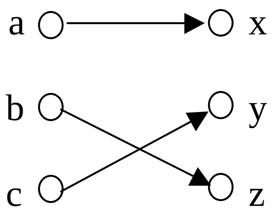
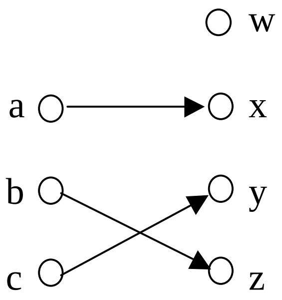


Example



f(a) = x

the image of c is y

the preimage of z is b

f({a,b}) = {x,z}

the domain of f is {a,b,c}

the codomain of f is {w,x,y,z}

the range of f is {x,y,z}

2. A function f : A→B is called <u>one-to-one</u>, or <u>injective</u>, if each element of B appears at most once as the image of an element of A.

Example Draw a function from {a,c} to {x,y,z} that is not 1-1.

Example Let A = {1,2,3,4,5} and B = {a,b,c,d,e,f}. How many 1-1 functions are possible from A to B?

3. Proving Functions Are 1-1

To prove that functions are 1-1 when the domain and codomain are infinite one assumes that $f(x) = f(y)$ and tries to deduce $x = y$.

Example Show that $f:R \rightarrow R$ where $f(x) = 5x + 2$ is 1-1.

$f(x) = f(y)$	Assumption
$5x + 2 = 5y + 2$	By definition of f
$5x = 5y$	Subtract 2
$x = y$	Divide by 5
f is 1-1	def of 1-1

4. Proving Functions Are Not 1-1

To prove that $f:A \rightarrow B$ is not 1-1 find two elements x and y in A, with $x \neq y$ and $f(x) = f(y)$.

Example Prove that $f:R \rightarrow R$ is not 1-1 where $f(x) = 2x^4 + 3x^2$

Solution $f(1) = f(-1) = 5$

Example Prove that $f:R \rightarrow R$ is not 1-1 where $f(x) = |x|$

Onto Functions

1. A function f:A → B is called <u>onto</u> or <u>surjective</u> if for all $b \in B$ there is at least one $a \in A$ with $f(a) = b$.

Note that f is onto if its range equals its codomain.

Example Draw a function from {a,b,c} to {x,y,z} that is not onto and one that is onto.

Example Let A = {1,2,3} and B = {a,b,c}. How many onto functions are possible from A to B?

Example Let A = {1,2,3,4} and B = {a,b,c}. How many onto functions are possible from A to B?

Example Let A = {1,2} and B = {a,b,c}. How many onto functions are possible from A to B?

2. If A has m elements and B has n elements then there

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

onto functions from A to B.

3. Proving Functions Are Onto

To prove that a function $f:A \to B$ is onto when the domain and codomain are infinite start with b in B and find an element a in A such that $f(a) = b$.

Accomplished three ways:

1. Find an inverse function by solving $f(a) = b$ for a, check that a is in A.

2. Make an educated guess for a in terms of b and demonstrate that $f(a) = b$ and check that a is in A.

3. Show the range = codomain

Example Prove that $f:R \to R$ where $f(x) = 5x + 2$ is onto.

Let r be any real.	Notation
Let $f(x) = r$	Set $f(x) = r$ to find x
$5x + 2 = r$	def of f
$x = (r - 2)/5$	solve for x
$(r - 2)/5$ is in R	Properties of reals
f is onto	def of onto

4. Proving Functions Are Not Onto

Find an element in B such that $f(a) \neq b$ for every a in A.

Example Show that $f:Z \to Z$ where $f(x) = x^3$ is not onto.

Example Show that $f:Z \to Z$ where $f(x) = 5x + 2$ is not onto.

Example Let A = B = R. Determine which are one-to-one and/or onto.

$f(x) = 2x$

$f(x) = x^2$

$f(x) = x^3$

$f(x) = | x |$

Example Repeat the above example with A = B = Z, the integers.

Special Functions in Discrete Math

1. The floor function denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to x. Its domain is R and range is Z.

$\lfloor 3.9 \rfloor = 3$ $\lfloor 3.1 \rfloor = 3$ $\lfloor -3.9 \rfloor = -4$

2. The ceiling function denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x. Its domain is R and range is Z.

$\lceil 3.9 \rceil = 4$ $\lceil 3.1 \rceil = 4$ $\lceil -3.9 \rceil = -3$

3. The number of ways in which it is possible to distribute m distinct objects into n identical containers, with no container empty is given by

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m.$$

S(m,n) is called a Stirling number of the second kind.

Example How many ways can a,b,c,d be distributed among three indistinguishable containers?

Solution Find S(4,3).

4. The number of ways in which it is possible to distribute m distinct objects into n non-identical containers, with no container empty is given by

$$n!S(m,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m .$$

This number gives the number of onto functions from a set a with m elements to a set B with n elements.

Example If A = {a,b,c,d} and B = {1,2,3} how many onto functions are there from A to B?

Solution 3! S(4,3) = 6(6) = 36

Example Let A = {a,b,c,d} and B = {v,w,x,y,z}. Determine the number of functions f:A → B where:

(a) f(A) = {v,x}

(b) |f(A)| = 2

Solution (a) 2!S(7,2)

(b) C(5,2) 2!S(7,2).

5. For any nonempty sets a, B, any function f: A× A → B is called a <u>binary operation</u> on A. If B is a subset of A then the operation is called <u>closed</u>.

6. A function g:A → A is called a <u>unary</u> operation on A.

Example (a) f : Z × Z → Z defined by f(a,b) = a − b is a closed binary operation on Z.

(b) If g : $Z^+ \times Z^+$ → Z defined by g(a,b) = a − b, then g is a binary operation on Z^+, but it is not closed.

(c) The function $h: Z \to Z$ defined by $f(a) = -a$ is a unary operation.

7. Let f be a binary operation on A:

f is commutative if $f(a,b) = f(b,a)$ for all a and b.

f is associative if $f(f(a,b),c) = f(a,f(b,c))$.

Example Let $h: Z \times Z \to Z$ defined by $h(a,b) = a|b|$.

(a) Determine if h is commutative.

(b) Determine if h is associative.

8. For sets A and B, if $D \subseteq A \times B$, then $\pi_A:D \to A$, defined by $\pi_A(a,b) = a$, is called projection onto the first coordinate. The function $\pi_B:D \to B$, defined by $\pi_B(a,b) = b$, is called projection onto the second coordinate.

Example Let $A = B = \mathbf{R}$ and consider the set $D = \{(x,y): y = x^2\}$. Then D is the set of points on the parabola $y = x^2$.

The point (3,9) is an element of D. $\pi_A(3,9) = 3$ and $\pi_B(3,9) = 9$.

$\pi_A(D) = \mathbf{R}$ so is π_A onto

$\pi_B(D) = \mathbf{R}^+ \cup \{0\} \neq \mathbf{R}$, so π_B is not onto.

The Pigeonhole Principle

1. The <u>Pigeonhole Principle</u> If m or more pigeons are placed into n pigeonholes and m > n, then there is at least one box containing two or more pigeons.

(If m or more objects are placed into n sets and m> n, then at least one set contains 2 or more objects.)

Example Among any group of 13 people, there must be at least 2 with a birthday in the same month.

Example Among any 7 games between the Yankees and the Mets, there must be one team who has won 4 or more games.

Example What is the minimum number of games that must be played between two teams to guarantee that one team will have won at least 10 games?

2. <u>Generalized Pigeonhole Principle</u> If N pigeons are placed into K pigeonholes, then at least 1 pigeonhole contains at least $\left\lceil \frac{N}{K} \right\rceil$ pigeons.

Example Among 100 people, there must be at least $\left\lceil \frac{100}{12} \right\rceil = 9$ who were born in the same month.

Example What is the minimum number of people required in a group to guarantee that at least 5 have a birthday in the same month?

Solution Let x be the minimum number. Then x must satisfy $\left\lceil \frac{x}{12} \right\rceil$ = 5 and $\left\lceil \frac{x-1}{12} \right\rceil$ = 4. If we solve $\frac{x}{12}$ = 5, we get x = 60. But notice that x could be any integer from 49, 50, ..., 60. Solving $\frac{x-1}{12}$ = 4 gives x = 49.

Example What is the minimum number of US residents required in a group to guarantee that at least 30 come from the same state?

Example Randomly select 7 integers from {1,2,...,10} and find two pairs in your selected set that sum to 11.

Example (a) Show that if 7 integers are selected from the {1,2,...,10}, there must be at least two pairs of these integers with sum 11.

(b) Show that the result in (a) is not true if 6 integers are selected.

Solution Consider as pigeonholes the 5 subsets {1,10}, {2,9}, {3,8}, {4,7}, {5,6}.

By the pigeonhole principle, if we place 7 integers into the 5 of these "pigeonholes", then at least 1 subset contains a pair of integers. This pair must sum to 11.

Now remove the 2 integers that add to 11, and remove the corresponding pigeonhole. This leaves 5 integers to be placed into 4 pigeonholes. By the pigeonhole principle again, there must be another pair of integers that sum to 11.

Note, the result is not true with the 6 integers 1,2,3,4,5,6.

Example Assume that in a group of 6 people every pair of individuals are either friends or enemies. Show that there are either 3 mutual friends or 3 mutual enemies.

Solution Let A be one of the 6 people, and represent the 6 people using vertices.

Join 2 vertices with a green line if they correspond to friends and use a blue line if they are enemies.

Of the 5 people other than A, there must be either 3 or more who are friends with A, or 3 or more who are enemies with A.

Why?

Suppose 3 of the other 5 people are friends of A, say B, C and D.

If any two of B, C and D are friends, then these 2 together with A form a group of 3 mutual friends.

If none of B, C and D are friends, then these 3 are mutual enemies.

The argument is similar if 3 of the 5 people excluding A are enemies of A.

Example Show that the result of the above example is not true for 5 people.

Example Wilma operates a computer with a magnetic tape drive. One day she is given a tape that contains 500,000 "words" of four or fewer lowercase letters. Can it be that the 500,000 words are all distinct?

Example Let S be a set of 6 positive integers whose maximum is 14. Show that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

Given a nonempty subset A of S, let S_A denote the sum of the elements in A.

Then $1 \leq S_A$ and $S_A \leq 9 + 10 + 11 + 12 + 13 + 14 = 69$.

We also know that there are $2^6 - 1 = 63$ nonempty subsets of S.

Consider the subsets A of S with 5 elements or less.

Now $1 \leq S_A$ and $S_A \leq 10 + 11 + 12 + 13 + 14 = 60$.

The number of subsets with 5 elements or less is 62.

By the pigeonhole principle there must be two subsets with the same sum.

Recurrence Relations

1. For $n \geq 0$ let a(n) denote a function expressed as a_n, where a_n is defined in terms of $a_{n-1}, a_{n-2}, ..., a_1, a_0$. Such a function is called a recurrence relation.

2. A geometric progression is an infinite sequence of terms where each term is obtained from the previous one by multiplying by the same number.

For example 5, 15, 45, 135, ... is a geometric progression.

It is defined by $a_{n+1} = 3a_n$, for $n \geq 0$, and $a_0 = 5$.

What is a_{10}?

3. Given a recurrence relation the solution to the relation is an explicit function that depends only on n and not on $a_{n-1}, a_{n-2}, ..., a_1, a_0$.

4. The unique solution of the recurrence relation $a_{n+1} = da_n$, where $n \geq 0$, and $a_0 = A$, is given by $a_n = Ad^n$.

Example Solve the recurrence relation $a_{n+1} = 7a_n$, where $n \geq 1$ and $a_2 = 98$.

Since $a_2 = 98$ and $a_{n+1} = 7a_n$ we know that $98 = 7a_1$. Therefore $a_1 = 14$. Similarly we find $a_0 = 2$.

So $a_n = Ad^n = 2(7^n)$.

Example Suppose that $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ and $a_0 = 2$.

(a) Find a_1 and a_2.
(b) Find a solution.
(c) Find a_{12}.

To find a solution we let $b_n = a_n^2$. Substituting gives $b_{n+1} = 5b_n$ and $b_0 = 4$.

The solution to this is $b_n = 4(5^n)$.

Since $a_n = \sqrt{b_n}$ we have $a_n = 2(\sqrt{5})^n$.

5. Recurrence relations of the form $a_{n+1} - da_n = 0$ are called <u>homogeneous</u>.

The relation is <u>nonhomogeneous</u> if the right hand side is equal to some nonzero function.

Example The number of comparisons needed to sort n numbers gives the following recurrence relation: $a_n = a_{n-1} + (n-1)$, for $n \geq 2$, and $a_1 = 0$.

(a) Find a_4.
(b) Find a solution for this relation.

Example Consider recurrence relation: $a_n + a_{n-1} - 6a_{n-2} = 0$, for $n \geq 2$, and $a_0 = -1$ and $a_1 = 8$.

(a) Find a_2 and a_3.
(b) How can we solve this?

7. Consider a recurrence relation of the form

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0, \; n \geq 2.$$

Suppose that $a_n = cr^n$.

Substituting we get $\quad C_0\, cr^n + C_1\, cr^{n-1} + C_2\, cr^{n-2} = 0$

Divide by cr^{n-2} to get $\quad C_0 r^2 + C_1 r + C_2 = 0$

This is called the characteristic equation. We can solve it by factoring or using the quadratic formula to determine general solutions (i.e. values for r).

c can be determined from initial conditions.

Example Find the characteristic equation for $a_n + a_{n-1} - 6a_{n-2} = 0$ and find general solutions.

Example Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$ and $F_0 = 0$, $F_1 = 1$.

Let $F_n = cr^n$.

Substituting gives $cr^{n+2} = cr^{n+1} + cr^n$.

The characteristic equation is $r^2 - r - 1 = 0$.

Using the Quadratic Formula we get roots $r = \frac{1 \pm \sqrt{5}}{2}$.

The general solution is $F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$.

$F_0 = 0$ gives the equation $0 = c_1 + c_2$.

$F_1 = 1$ gives the equation $1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$.

A little algebra gives $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$.

So $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$.

Note $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ is called the golden ratio.

Example Check the above solution for $n = 2$.

Example Find the general solution for $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, $n \ge 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$.

Let $a_n = cr^n$.

Substituting gives $2cr^{n+3} = cr^{n+2} + 2cr^{n+1} - cr^n$

The characteristic equation is $2r^3 - r^2 - 2r + 1 = 0$.

Factoring we get $(2r - 1)(r - 1)(r + 1) = 0$

The roots are 1/2, 1 and -1.

The general solution is $a_n = c_1(1)^n + c_2(-1)^n + c_3(1/2)^n$

$$= c_1 + c_2(-1)^n + c_3(1/2)^n$$

Using the initial conditions $a_0 = 0$, $a_1 = 1$, $a_2 = 2$ we get the equations:

$0 = c_1 + c_2 + c_3$ $\quad$ $1 = c_1 - c_2 + (1/2)c_3$ $\quad$ $2 = c_1 + c_2 + (1/4)c_3$

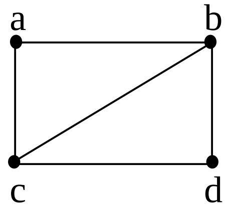
The particular solution is $a_n = (5/2) + (1/6)(-1)^n + (-8/3)(1/2)^n$

Check Calculate a_3 using the recurrence relation $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, and using the formula $a_n = (5/2) + (1/6)(-1)^n + (-8/3)(1/2)^n$

Example (Repeated roots) Find the solution for $a_{n+2} = 6a_{n+1} - 9a_n$, $n \geq 2$, $a_0 = 1$, $a_1 = 6$.

Introduction to Graphs

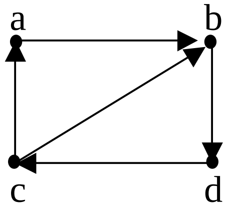
1. An undirected graph G = (V,E) consists of a set of vertices V along with a set of edges E, such that each edge consists of a pair of vertices.



V = {a,b,c,d}

E = { {a,b}, {a,c}, {b,c}, {b,d}, {c,d} }

A directed graph G = (V,E) consists of a set of vertices V along with a set of directed edges E, such that each edge consists of a pair of an ordered pair vertices.



V = {a,b,c,d}

E = { (a,b), (c,a), (b,c), (b,d), (d,c), (a,a) }

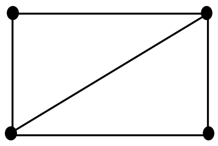
The edge from a to a is called a loop.

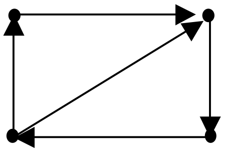
2. Two vertices are called adjacent if they are joined by an edge. If x and y are adjacent then we denote the edge joining them by {x,y} in an undirected graph, and (x,y) if there is an edge from x to y.

3. The degree of a vertex x in a undirected is the number of vertices that are adjacent to x, denoted by deg x.

In a directed graph we say the vertex x has <u>in-degree</u> equal to p is there are p edges going into p, and out-degree equal to q if there are q edges going out of x

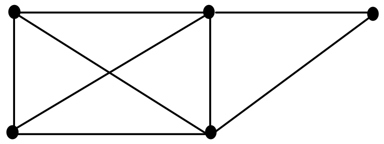
Example Determine the degrees of all vertices in the graphs below.



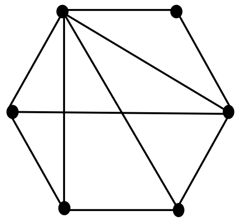


Example For each graph determine the degrees of every vertex, the sum of all degrees and the number of edges.

(a)



(b)



4. **Theorem** Given any graph G = (V,E), $\sum_{x \in V} \deg(x) = 2|E|$.

(the sum of the vertex degrees equals twice the number of edges)

Proof ?

Example Is it possible to have a group of 7 computers where each computer is connected to exactly 3 of the other computers?

5. Let x and y be vertices in a undirected graph G. An <u>x-y walk</u> in G is a sequence of vertices and edges $x = x_0, e_1, x_2, e_2, ..., e_n, x_n = y$. The length of the walk is n.

6. A graph is called <u>complete</u> if there is an edge joining every pair of vertices. The complete graph with n vertices is denoted by K_n.

Example Draw K_2, K_3, K_4 and K_5.

Example Find a formula in terms of n that gives the degree of every vertex in K_n, and one for n the number of edges in K_n.

Example In the beginning of every session of the US Supreme Court the 9 justices all shake hands. How many handshakes take place?

7. A path is a an x-y walk where no vertex occurs more than once.

8. A cycle is a path that begins and ends at the same vertex. A cycle with n edges is denoted by C_n, for $n > 2$.

Example Draw C_3, C_4 and C_5.

Example Determine the degree of every vertex in C_n and the number of edges in C_n.

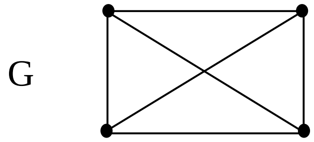
9. The n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the corresponding bit strings differ in exactly one bit.

Example Draw Q_1, Q_2 and Q_3. Label all vertices using bit strings.

Example Determine the degree of every vertex in Q_n and the number of edges in Q_n.

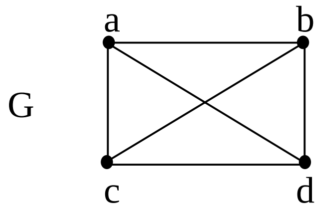
10. If G = (V,E) is a graph, W a subset of V, and F a subset of E, then H = (W,F) is called a <u>subgraph</u> of G.

Example How many subgraphs does G have containing 3 edges?



11. If W is a subset of vertices, then the subgraph <u>induced</u> by W is the subgraph with vertices W and all edges in G joining nodes in W.

Example Let G be the following graph.



Draw the subgraph induced by W = {a,b,c}.

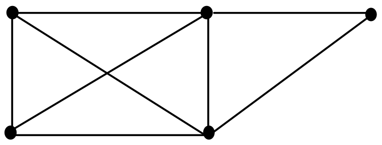
12. A graph is called connected if for every pair of vertices x and y there exists a path joining x to y.

13. A connected graph with no cycles is called a tree.

Example (a) Find a subgraph of G that contains 4 edges and is a tree.

(b) Find a subgraph of G that contains 4 edges and is not a tree.

G

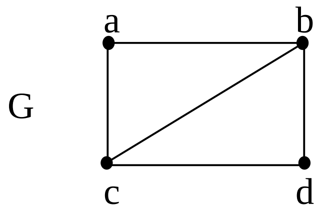


14. If a graph is not connected, then the connected subgraphs are called the components of the graph. κ(G) is used to denoted the number of components.

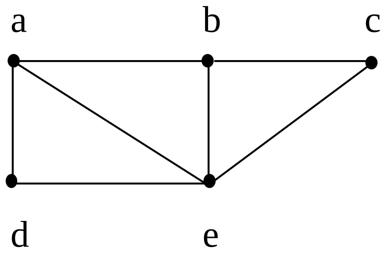
15. Given a graph G = (V,E) with n vertices, the <u>complement</u> of G, denoted $\overline{G}$ is the graph with vertices V, and all edge of K_n that are not in G.

Example Find the complement of the given graph.

(a)



(b)

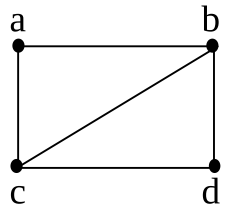


Graph Isomorphism

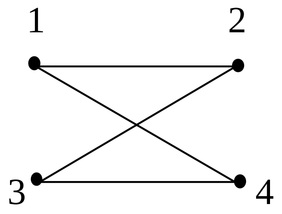
1. Let $G = (V,E)$ be a graph with vertices $V = \{v_1, \ldots , v_n\}$. The <u>adjacency matrix</u> A of G is the n by n zero-one matrix with $a_{ij} = 1$, if $\{v_i,v_j\}$ is an edge in G, and $a_{ij} = 0$, if $\{v_i,v_j\}$ is not an edge in G.

Note: the matrix depends on how the vertices are labeled.

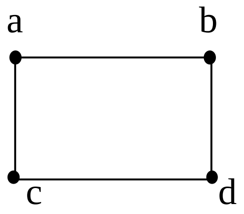
Example Find an adjacency matrix for the graph



What do the following graphs have in common ?



G_1



G_2

2. Two graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are called <u>isomorphic</u> if there is a one-to-one and onto function f from V_1 to V_2 such that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2.

To show that two graphs are isomorphic we need to find a one-to-one function f from V_1 to V_2 such that when the adjacency matrix of G_1 is labeled a,b,c,..., we get the same matrix as the adjacency matrix of G_1 using labels f(a), f(b), f(c),.., etc.

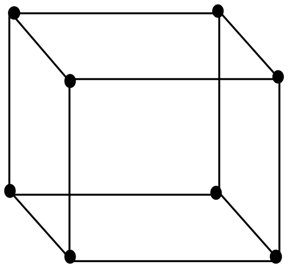
Example Show that the graphs G_1 and G_2 given above are isomorphic.

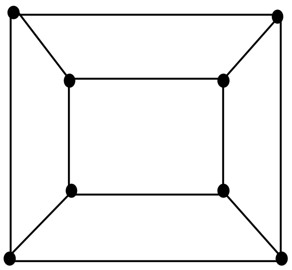
Vertices in G_1	(x)	1	2	3	4
Vertices in G_2	f(x)	a	b	d	c

Adjacency matrix for G_1 Adjacency matrix for G_2

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & a & b & d & c \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 \\ c & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

Example Show that the graphs G_1 and G_2 given below are isomorphic.





3. Non-isomorphic graphs.

We can show that two graphs are not isomorphic by showing that they do not both share a property that isomorphic graphs must share. These properties are called <u>invariant</u> properties.

If G and H are isomorphic graphs, then:

(i) G and H must have the same number of vertices and edges.

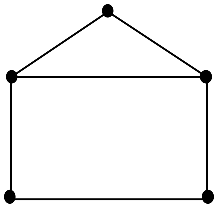
(ii) The set of vertex degrees of G must be the same as the set of vertex degrees of H.

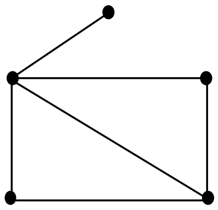
(iii) The subgraphs of G must be isomorphic to corresponding subgraphs of H.

(iv) The complements of G and H are isomorphic.

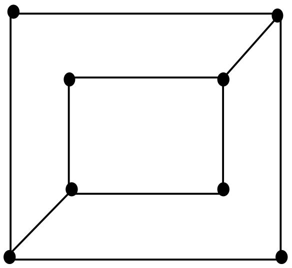
Example Show that the following pairs of graphs are not isomorphic. Use the above invariant properties.

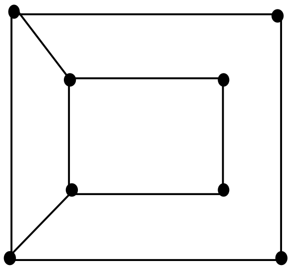
(a)





(b)





Instant Insanity

The game of instant insanity is played with four cubes. Each of the six faces of the cube is painted with one of 4 colors R, W, B or Y. The object of the game is to place the cubes in a column of 4 such that all 4 different colors appear on each of the 4 sides of the column.

To model the puzzle we construct a graph as follows.

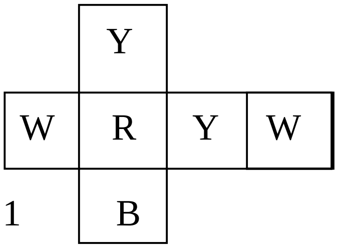
First, label each cube using 1, 2, 3 and 4.

Create 4 vertices labeled R, W, B and Y.

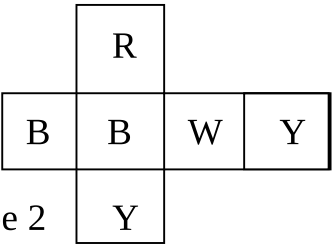
Examine each cube and determine the colors on opposite faces. For each pair of opposite faces place an edge in the graph joining the two colors of the opposite faces. Label the edge with the label used for the cube.

For example if cube 1 has opposite faces B and Y, place an edge in the graph joining the b and Y vertices and label the edge with a 1.

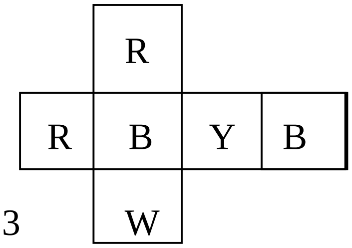
Example Construct the graph for the following instant insanity puzzle.



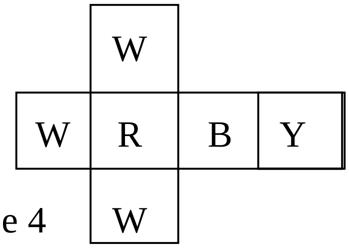
Cube 1



Cube 2



Cube 3



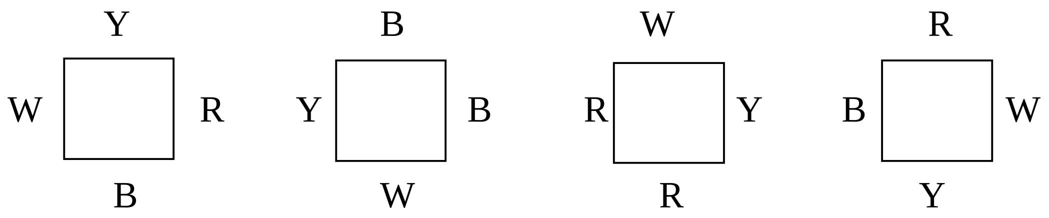
Cube 4

Theorem An Instant Insanity Puzzle has a solution if and only if its corresponding labeled graph has two disjoint subgraphs consisting of 4 edges, such that every vertex has degree 2, and every edge has a different label.

The subgraphs can be either a cycle of length 4, two cycles of length 2, a cycle of length 3 and a loop, a cycle of length 2 and two loops, or 4 loops.

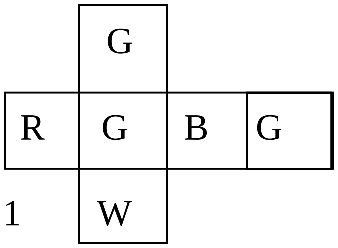
The solution to the above problem is obtained from the following subgraphs.

The solution is shown below.

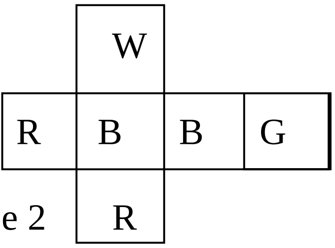


Cube 1 Cube 2 Cube 3 Cube 4

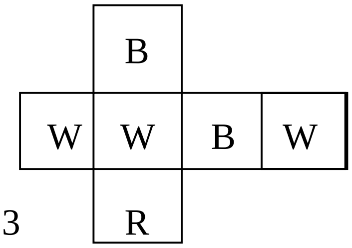
Example Solve the Instant Insanity Puzzle given below.



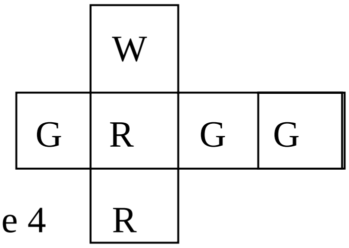
Cube 1



Cube 2



Cube 3



Cube 4

Planar Graphs

1. A graph G is called planar if G can be drawn in the plane with its edges intersecting only at vertices of G.

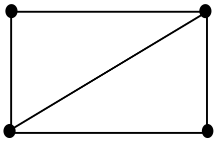
Example Draw K_4 as a planar graph. Can K_5 be drawn as a planar graph?

2. A graph $G = (V,E)$ is called bipartite if $V = V_1 \cup V_2$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$ every edge joins a vertex in V_1 to a vertex in V_2.

Example Which of the following are bipartite?

(a)

(b)



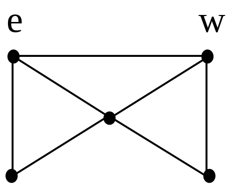
3. **Theorem** A graph G is bipartite if and only if G does not contain an odd cycle.

4. A bipartite graph $G = (V,E)$ with $V = V_1 \cup V_2$ is called complete if there is an edge joining every vertex in V_1 to every vertex in V_2. If V_1 has m vertices and V_2 has n vertices, the complete bipartite graph is denoted by $K_{m,n}$.

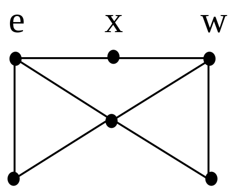
Example Draw $K_{3,3}$ and $K_{2,4}$.

5. Let $G = (V,E)$ be a loop-free graph with $E \neq \emptyset$. An <u>elementary subdivision</u> of G results when an edge $e = \{u,w\}$ is removed from G and then the edges $\{u,x\}$ and $\{x,w\}$ are add to $G - e$, where $x \notin V$.

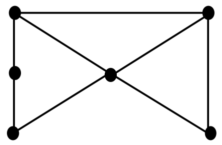
Example



G_1



G_2



G_3

G_2 is obtained from G_1 by an elementary subdivision.

G_3 is obtained from G_1 by an elementary subdivision.

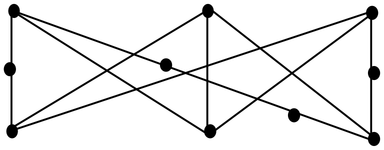
G_3 can not be obtained from G_2 by an elementary subdivision.

6. The loop-free graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are called <u>homeomorphic</u> if they are isomorphic or if they can be obtained from the same loop-free graph H by a sequence of elementary subdivisions.

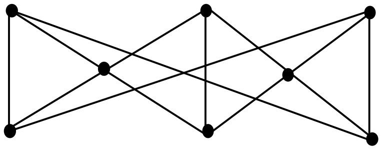
7. If two graphs are homeomorphic, then they are either both planar or both nonplanar.

Example Determine if the given graph is homeomorphic to $K_{3,3}$.

(a)

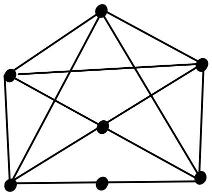


(b)

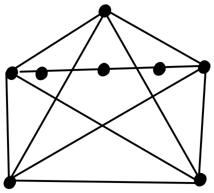


Example Determine if the given graph is homeomorphic to K_5.

(a)



(b)



8. **Kuratowski's Theorem** A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

Example Show that there is a subgraph of the Petersen graph that is homeomorphic to $K_{3,3}$.

Example Consider the following planar graphs and solids. Let v be the number of vertices, e the number of edges, and r the number of regions. Determine v, e and r for each graph/solid.

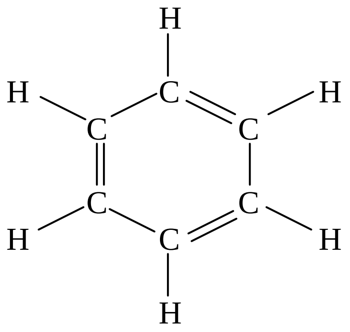
Object	v	e	r
K_3			
K_4			
cube			
tetrahedron			
octahedron			
icosahedron			
dodecahedron			

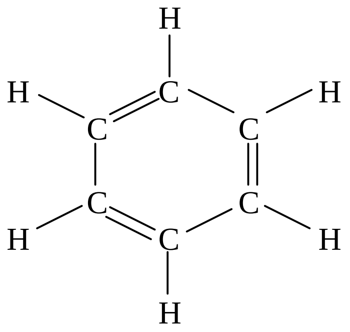
Example Is there a relationship among v, e and r?

9. E**uler's Theorem For Planar Graphs** Let $G = (V,E)$ be a connected planar graph. Then $v - e + r = 2$.

Counting Perfect Matchings in Hexagonal Systems

1. The German chemist August Kekulé (1829-1896) discovered the molecular structure of benzene after he dreamed of a snake swallowing its own tail.



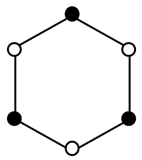


Kekulé’s discovery initiated the study of special types of graphs used to model benzene-like molecules called *benzenoids*.

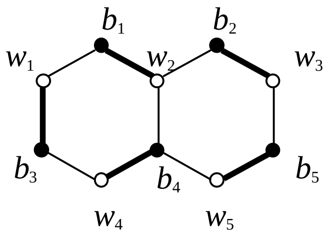
2. A graph G is called 2-*connected* if it is connected and at least 2 vertices must be removed to make G disconnected.

A *hexagonal system* is a 2-connected planar graph such that each interior face can be drawn as a regular hexagon. These graphs are also bipartite.

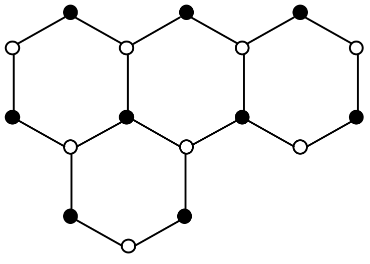
Given a graph G, a *perfect matching* M in G is a subgraph containing all the vertices of G such that every vertex has degree 1; the number of perfect matchings in G is denoted by $\Phi(G)$.



H_1



H_2



H_3

$\Phi(H_1) = 2$, $\Phi(H_2) = 3$ and $\Phi(H_3) = 0$

3. Given a molecular model of a benzenoid, its corresponding hexagonal system *H* is obtained by removing the edges representing carbon-hydrogen bonds and letting the remaining edges of *H* represent either single or double carbon-carbon bonds.

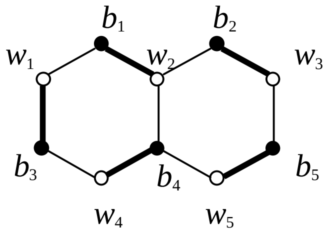
All hexagonal systems that arise from benzenoids have perfect matchings, and each perfect matching is a possible location for all the double carbon-carbon bonds.

Conversely, for each hexagonal system containing a perfect matching, experimental chemistry tells us that a benzenoid may be synthesized.

Therefore, chemists are interested in the following question. Given a hexagonal system *H*, is $\Phi(H) \neq 0$?

Chemical properties of a benzenoid such as stability and energy levels depend on the number of perfect matchings in its corresponding hexagonal system, so chemists seek efficient methods to calculate $\Phi(H)$.

4. Define the $n \times n$ *biadjacency matrix* $A(H) = [a_{ij}]$, by $a_{ij} = 1$ if $\{b_i, w_j\} \in E$ and $a_{ij} = 0$ if $\{b_i, w_j\} \notin E$.



H_2

$$\begin{array}{c|ccccc} & w_1 & w_2 & w_3 & w_4 & w_5 \\ \hline b_1 & 1 & 1 & 0 & 0 & 0 \\ b_2 & 0 & 1 & 1 & 0 & 0 \\ b_3 & 1 & 0 & 0 & 1 & 0 \\ b_4 & 0 & 1 & 0 & 1 & 1 \\ b_5 & 0 & 0 & 1 & 0 & 1 \end{array} \qquad (1)$$

5. **Theorem** *For a hexagonal system H,* $\Phi(H) = |\det(A(H))|$.

Proof Every perfect matching in H corresponds to a permutation of

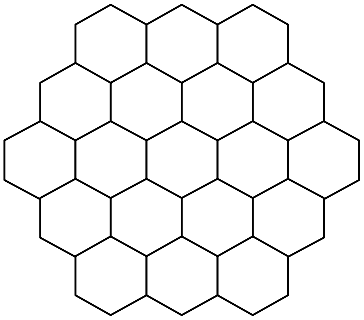
$\{1,2, \ldots ,n\}$.

There is a one-to-one correspondence between the nonzero terms in the expansion of the determinant of $A(H)$ and the perfect matchings in H.

Since $a_{ij} = 0$ or 1 in $A(H)$, the nonzero terms in the expansion of $\det(A(H))$ are all either 1 or -1.

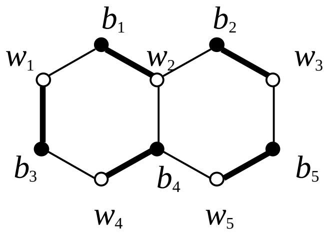
$|\det(A(H))|$ may be used to compute $\Phi(H)$ if we know that all the nonzero terms corresponding to perfect matchings in H have the same sign. Lemmas 1 and 2 imply this, and hence prove Theorem.

Example



A hexagonal system with 980 perfect matchings.

Example Find the determinant of the matrix below.



H_2

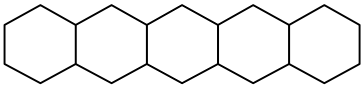
$$\begin{array}{c|ccccc|} & w_1 & w_2 & w_3 & w_4 & w_5 \\ b_1 & 1 & 1 & 0 & 0 & 0 \\ b_2 & 0 & 1 & 1 & 0 & 0 \\ b_3 & 1 & 0 & 0 & 1 & 0 \\ b_4 & 0 & 1 & 0 & 1 & 1 \\ b_5 & 0 & 0 & 1 & 0 & 1 \end{array}$$

Lemma 1. *Let H be a hexagonal system and let M_1 and M_2 be perfect matchings in H. Then every cycle C in $M_1 \cup M_2$ satisfies* $|C| \equiv 2 \bmod 4$.

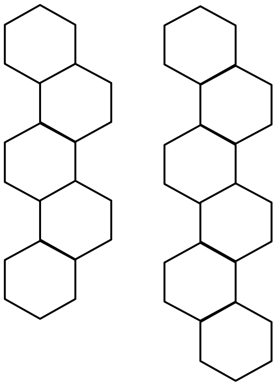
Let H be a hexagonal system, M a perfect matching in H, and σ the permutation in the symmetric group S_n corresponding to M. Define the *sign* of M to be +1 if σ is an even permutation and –1 if σ is an odd permutation.

Lemma 2. *For a hexagonal system H, all perfect matchings in H have the same sign.*

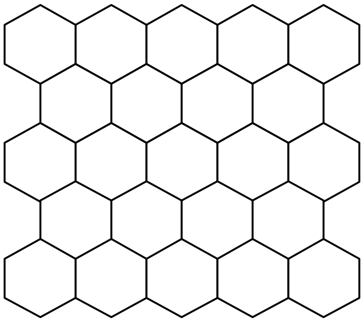
Special Classes of Hexagonal Systems



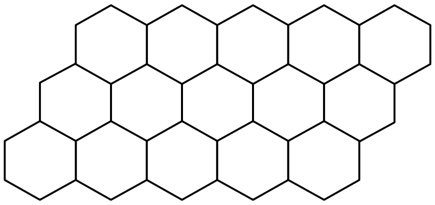
The linear chain P[5,1]



Fibonaccene chains



The Rectangular Hexagonal System *R*[5,3]

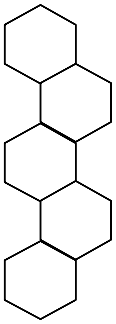


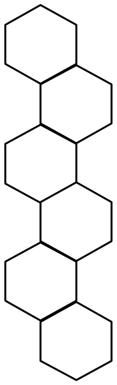
The Parallelogram Hexagonal System *P*[5,3]

6. A hexagonal system is called a *fibonaccene chain* if it consists of a chain of hexagons $H_1, \ldots, H_p$, with H_1 on top, and only the following shared edges: For i even and $1 < i < p$, H_i shares its northwest edge with H_{i-1} and its southwest edge with H_{i+1}; H_1 and H_p share edges as indicated in the figure.

Theorem. *Let H be a fibonaccene chain with h hexagons and let* $a_h = \Phi(H)$. *Then for* $h \geq 2$, a_h *satisfies* $a_h = a_{h-1} + a_{h-2}$, *where* $a_0 = 1$ *and* $a_1 = 2$.

Example Determine the number of perfect matchings in the chains given below.





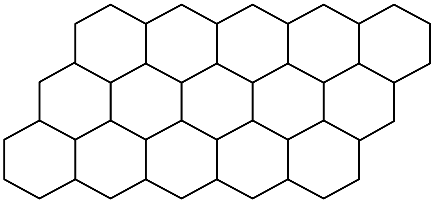
Solution The given graphs are fibonaccene chains with 5 and 6 hexagons. By the Theorem, we may calculate Φ using the Fibonacci sequence 1, 2, 3, 5, 8, 13, 21; here $a_5 = 13$ and $a_6 = 21$.

7. A hexagonal system consisting of h hexagons such that all adjacent pairs of hexagons share exactly one vertical edge and no nonvertical edges is called a *linear chain* of length h.

A *parallelogram hexagonal system,* denoted by $P[h,p]$, consists of p linear chains $L_1, \ldots, L_p$, each of length h, such that for $i = 1, \ldots, p - 1$, all southern edges of hexagons in L_i, except for the southeast edge of the eastern-most hexagon in L_i, are also northern edges of hexagons in L_{i+1}, except for the northwest edge of the western-most hexagon in L_{i+1}.

Theorem *For a parallelogram hexagonal system* $P[h,p]$, *we have* $\Phi(P[h,p]) = C(h + p,p)$.

Example Determine the number of perfect matchings in the following graph.



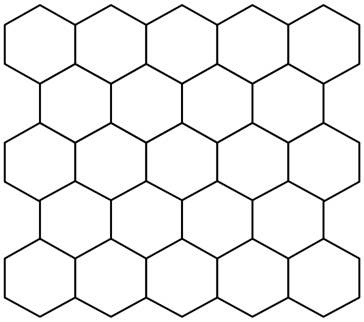
Solution C(8,3) = 56

8. A *rectangular hexagonal system*, denoted by $R[h,p]$, consists of p linear chains, $L_1, \ldots, L_p$, of length h, together with $p-1$ linear chains $\overline{L}_1, \ldots, \overline{L}_{p-1}$ of length $h-1$, such that for $i = 1, \ldots p-1$, all northern edges of $\overline{L}_i$ are southern edges of L_i, and all southern edges of $\overline{L}_i$ are northern edges of L_{i+1}.

Theorem *For a rectangular hexagonal system* $R[h,p]$, *we have* $\Phi(R[h,p]) =$

$(h+1)^p$.

Example Determine the number of perfect matchings in the following graph.



Solution $\Phi(R[5,3]) = 6^3 = 216$.